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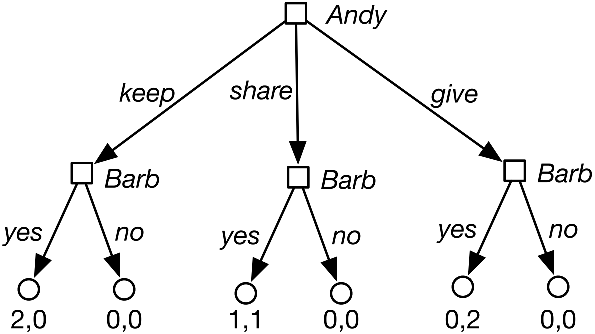
- Multi-agent reinforcement learning: each agent has its own Q function.

Fully-observable Game Tree Search

```
procedure GameTreeSearch(n)
    Inputs
        n a node in a game tree
    Output
        A pair: value for each agent for node n, path that gives this value
    if n is a leaf node then
        return {i : evaluate(i, n)}, None
    else if n is controlled by agent i then
        max := −∞
        for each child c of n do
            score, path := GameTreeSearch(c)
            if score[i] > max then
                max := score[i]
                res := (score, c : path)
        return res
```

Extensive Form of a Game

What happens with this game? Payoff is for *Andy*, *Barb*



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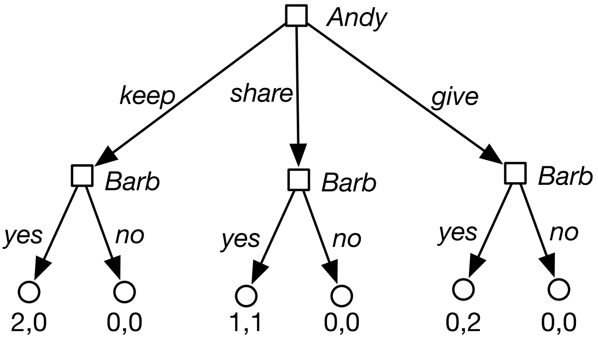

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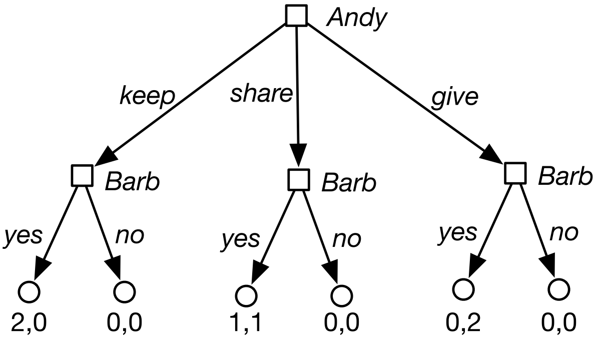

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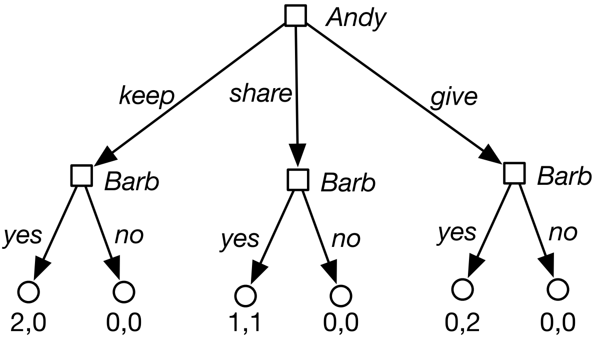

What if the 2,0 payoff was 1.9,0.1?
Should Barb be rational / predictable?
What should Andy do if Barb threatens to not do her best action?

Pruning Dominated Strategies

Special case: two person, competitive (zero sum) game. Utility for one agent is negative of utility of other agent. $\Longrightarrow$ minimax.

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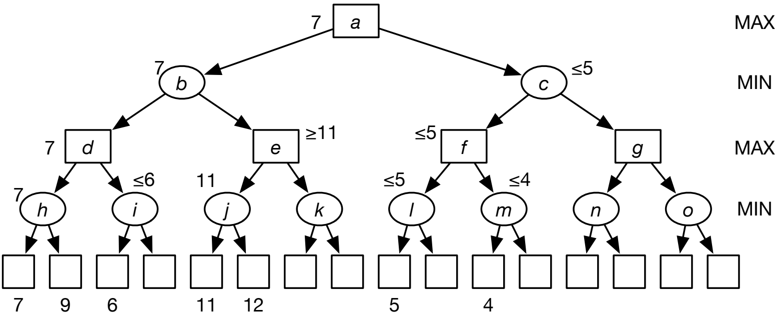

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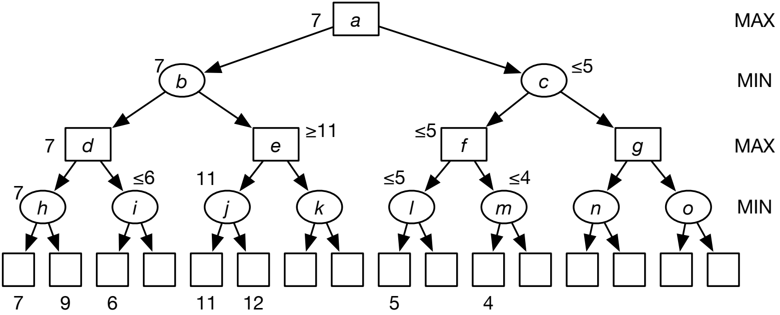
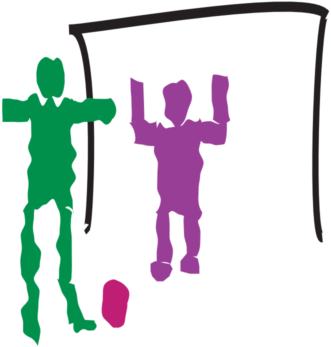


- square MAX nodes controlled by maximizing agent score
- round MIN nodes are controlled by a minimizing adversary
- leaves without a number do not need to be evaluated.

  $\longrightarrow$ α-β pruning.

Partial Observability and Competition



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Probability of a goal.

- Each agent decides what to do without seeing the other agent's action.

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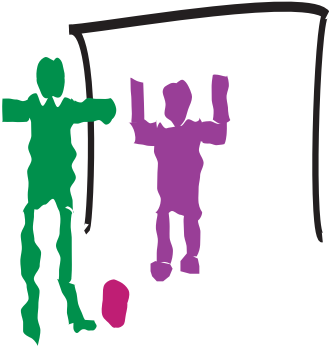

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- What should each agent do?

Strategy Profiles

- Assume an n-player game in normal form
- A strategy for an agent is a probability distribution over the actions for this agent.

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- A strategy for an agent is a probability distribution over the actions for this agent.
- A strategy profile is an assignment of a strategy to each agent.
- A strategy profile σ has a utility for each agent.
  Let $utility(\sigma, i)$ be the utility of strategy profile σ for agent i.
- If σ is a strategy profile:
  σ_i is the strategy of agent i in σ,
  σ_{-i} is the set of strategies of the other agents.
  Thus σ is $\sigma_i\sigma_{-i}$

Nash Equilibria

- σ_i is a best response to σ_{-i} if for all other strategies σ_i' for agent i,

$$utility(\sigma_i\sigma_{-i}, i) \geq utility(\sigma_i'\sigma_{-i}, i).$$

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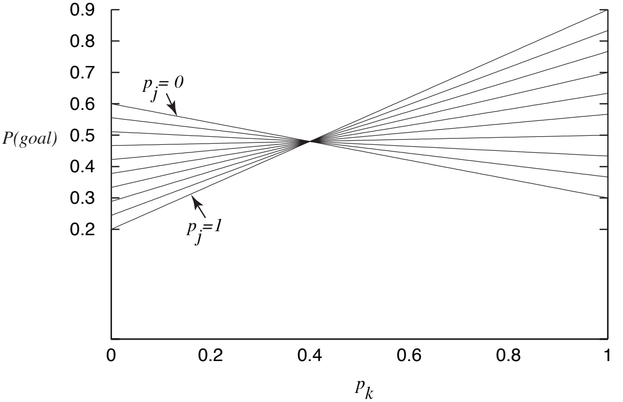
- Theorem [Nash, 1950] Every finite game has at least one Nash equilibrium.



		goalkeeper left	goalkeeper right
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Probability of a goal.

p_k is $P(kicker = right)$
p_j is $P(goalkeeper = right)$

Multiple Equilibria

Hawk-Dove Game:

		Agent 2	
		dove	hawk
Agent 1	dove	R/2,R/2	0,R
	hawk	R,0	-D,-D

D and R are both positive with $D >> R$.

Coordination

Just because you know the Nash equilibria doesn't mean you know what to do:

		Agent 2	
		shopping	football
Agent 1	shopping	2,1	0,0
	football	0,0	1,2

Prisoner's Dilemma

Two strangers are in a game show. They each have the choice:

- Take $100 for yourself
- Give $1000 to the other player

This can be depicted as the playoff matrix:

		Player 2	
		take	give
Player 1	take	100,100	1100,0
	give	0,1100	1000,1000

Tragedy of the Commons

Example:

- There are 100 agents.
- There is an common environment that is shared amongst all agents. Each agent has 1/100 of the shared environment.
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  $1000 - 10000 = -9000$

Computing Nash Equilibria

To compute a Nash equilibria for a game in strategic form:

- Eliminate dominated strategies
- Determine which actions will have non-zero probabilities. This is the support set.
- Determine the probability for the actions in the support set

Eliminating Dominated Strategies

		Agent 2		
		d_2	e_2	f_2
	a_1	3,5	5,1	1,2
Agent 1	b_1	1,1	2,9	6,4
	c_1	2,6	4,7	0,8

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Given a support set:

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Search over support sets to find a Nash equilibrium

Example: computing Nash equilibrium

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Similarly for goal keeper: $P(jump\ right) = 0.3$
Probability of a goal is:
$(0.6*0.7)*0.6+(0.6*0.3)*0.2+(0.4*0.7)*0.3+(0.4*0.3)*0.9 = 0.48$

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- Assuming those statistics reflect the stochastic policy of the other agent, play a best response.
- Both players using fictitious play converges to a Nash equilibrium for many types of games (including two-player zero-sum games).
- If an opposing agent knew the exact strategy (whether learning or not) agent A was using, and could predict what agent A would do, it could exploit that knowledge.

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  Inputs

    S is states, A is actions, α is step size, γ discount

    q_init and p_init (> 0) are initial Q and P values

  Local

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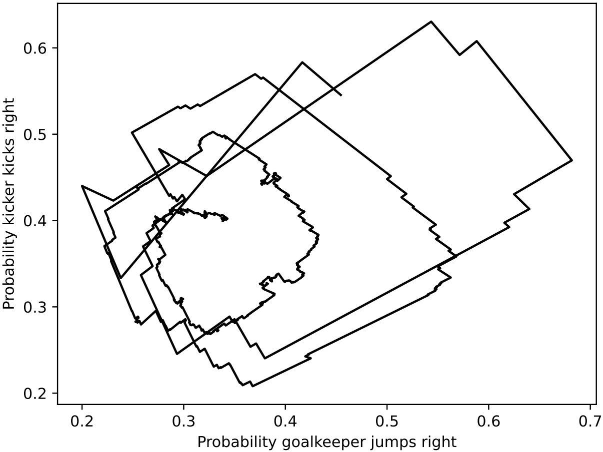
    $s := s'$; $a := a'$
  until termination

Stochastic Policies



Repeated playing goal-kick game with single state ($\alpha = 0.1$, $\gamma = 0$, $q\_init = 1$, $p\_init = 5$).
AIPython: `masLearn.py`

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- It was trained on self-play, playing itself for tens of millions of games.