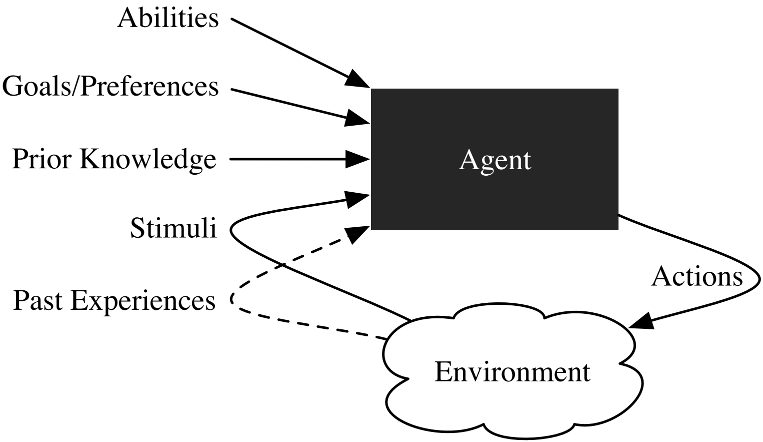
Agents acting in an environment: inputs and output



Goals and Preferences

Alice . . . went on "Would you please tell me, please, which way I ought to go from here?"
"That depends a good deal on where you want to get to," said the Cat.
"I don't much care where —" said Alice.
"Then it doesn't matter which way you go," said the Cat.

Lewis Carroll, 1832–1898
Alice's Adventures in Wonderland, 1865
Chapter 6

Learning Objectives

At the end of the class you should be able to:

- justify the use and semantics of utility
- know the assumptions behind measures of preference
- estimate the utility of an outcome

Preferences

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- Agents have preferences over outcomes
- A rational agent will do the action that has the best outcome for them
- Sometimes agents don't know the outcomes of the actions, but they still need to compare actions
- Agents have to act.
  (Doing nothing is (often) an action).

Preferences Over Outcomes

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- $o_1 \sim o_2$ means $o_1 \succeq o_2$ and $o_2 \succeq o_1$.
- $o_1 \succ o_2$ means $o_1 \succeq o_2$ and $o_2 \not\succeq o_1$

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- When we talk about outcomes, we will include lotteries.

Properties of Preferences

- Completeness: Agents have to act, so they must have preferences:

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- Transitivity: Preferences must be transitive:

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(Similarly for other mixtures of $\succ$ and $\succeq$.)

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(Similarly for other mixtures of $\succ$ and $\succeq$.)
Rationale: otherwise $o_1 \succeq o_2$ and $o_2 \succ o_3$ and $o_3 \succeq o_1$.
If they are prepared to pay to get o_2 instead of o_3,
and are happy to have o_1 instead of o_2,
and are happy to have o_3 instead of o_1
$\longrightarrow$ money pump.

Properties of Preferences (cont.)

Monotonicity: An agent prefers a larger chance of getting a better outcome than a smaller chance:

- If $o_1 \succ o_2$ and $p > q$ then

$$[p : o_1, 1 - p : o_2] \succ [q : o_1, 1 - q : o_2]$$

Consequence of axioms

- Suppose $o_1 \succ o_2$ and $o_2 \succ o_3$. Consider whether the agent would prefer
  - o_2
  - the lottery $[p : o_1, 1 - p : o_3]$

  for different values of $p \in [0, 1]$.
- Plot which one is preferred as a function of p:

Properties of Preferences (cont.)

Continuity: Suppose $o_1 \succ o_2$ and $o_2 \succ o_3$, then there exists a $p \in [0, 1]$ such that

$$o_2 \sim [p : o_1, 1 - p : o_3]$$

Properties of Preferences (cont.)

Decomposability: (no fun in gambling). An agent is indifferent between lotteries that have same probabilities and outcomes. This includes lotteries over lotteries. For example:

$$[p : o_1, 1-p : [q : o_2, 1-q : o_3]]$$
$$\sim \quad [p : o_1, (1-p)q : o_2, (1-p)(1-q) : o_3]$$

Properties of Preferences (cont.)

Substitutability: if $o_1 \sim o_2$ then the agent is indifferent between lotteries that only differ by o_1 and o_2:

$$[p : o_1, 1 - p : o_3] \sim [p : o_2, 1 - p : o_3]$$

Alternative Axiom for Substitutability

Substitutability: if $o_1 \succeq o_2$ then the agent weakly prefers lotteries that contain o_1 instead of o_2, everything else being equal.
That is, for any number p and outcome o_3:

$$[p : o_1, (1-p) : o_3] \succeq [p : o_2, (1-p) : o_3]$$

What we would like

- We would like a measure of preference that can be combined with probabilities. So that

$$\begin{aligned} & value([p : o_1, 1 - p : o_2]) \\ = \quad & p * value(o_1) + (1 - p) * value(o_2) \end{aligned}$$

- Money does not act like this.
  What would you prefer

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- It may seem that preferences are too complex and muti-faceted to be represented by single numbers.

Theorem

If preferences follow the preceding properties, then preferences can be measured by a function

$$utility : outcomes \rightarrow [0, 1]$$

such that

- $o_1 \succeq o_2$ if and only if $utility(o_1) \geq utility(o2)$.
- Utilities are linear with probabilities:

$$utility([p_1 : o_1, p_2 : o_2, \ldots, p_k : o_k]) = \sum_{i=1}^{k} p_i * utility(o_i)$$

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This exists by the Continuity property.

Proof (cont.)

- Suppose $o_1 \succeq o_2$ and $utility(o_i) = u_i$, then by Substitutability,

$$[u_1 : best, 1 - u_1 : worst]$$
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$$\begin{aligned} u = utility(\quad [\quad & p_1 : [u_1 : best, 1 - u_1 : worst] \\ & \ldots \\ & p_k : [u_k : best, 1 - u_k : worst]]) \end{aligned}$$

Proof (cont.)

- By decomposability, this is equivalent to:

$$u = utility(\; [\; p_1 u_1 + \cdots + p_k u_k : best,\; p_1(1-u_1) + \cdots + p_k(1-u_k) : worst]])$$

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- Thus, by definition of utility:

$$u = p_1 * u_1 + \cdots + p_k * u_k$$

Utilities

Two conditions of utility:

- $o_1 \succeq o_2$ if and only if $utility(o_1) \geq utility(o2)$.
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- Often a different scale, such as $[0, 100]$, is used for utility.
- Sometimes negative values – costs – are used.

Delivery Robot Decision

- The robot can choose to wear pads to protect itself or not.
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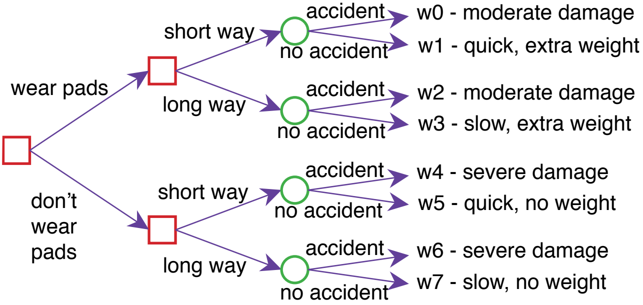

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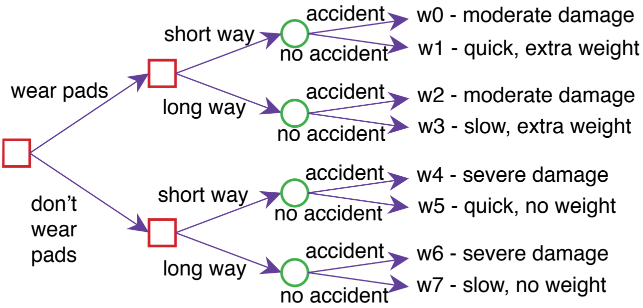
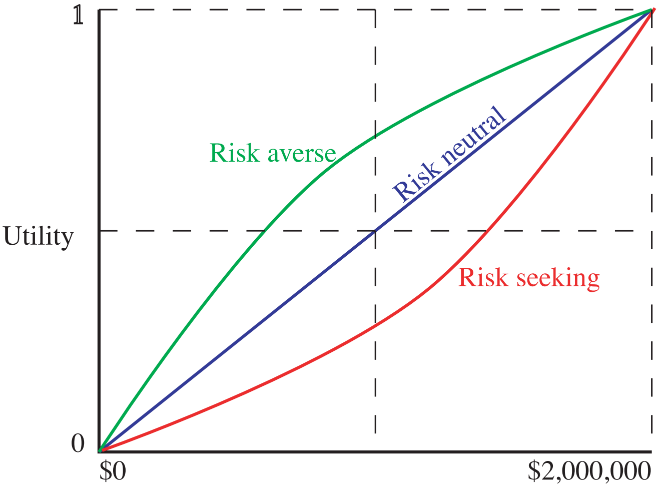


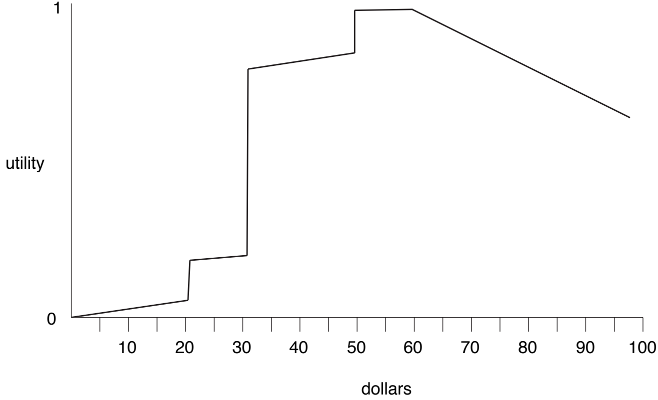
- What are reasonable utilities for the 8 outcomes $w0, \ldots, w7$? (suppose range $[0, 100]$)

Utility as a function of money



Possible utility as a function of money

Someone who really wants a toy worth $30, but who would also like one worth $20:



Factored Representation of Utility

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- Many ways to represent the same utility:
  — a number can be added to one factor as long as it is subtracted from others.

Additive Utility

- An additive utility has a canonical representation:

$$u(X_1, \ldots, X_n) = w_1 * u_1(X_1) + \cdots + w_n * u_n(X_n).$$

- If $best_i$ is the best value of X_i, $u_i(X_i{=}best_i) = 1$.
  If $worst_i$ is the worst value of X_i, $u_i(X_i{=}worst_i) = 0$.
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$$w_1 = u(best_1, x_2, \ldots, x_n) - u(worst_1, x_2, \ldots, x_n).$$

for any values $x_2, \ldots, x_n$ of $X_2, \ldots, X_n$.

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- Assumption behind additive utility: for all x_1, x_1',
  $u(x_1, x_2, \ldots, x_n) - u(x_1', x_2, \ldots, x_n)$ is the same for all
  $x_2, \ldots, x_n$, and similarly for other positions.

Complements and Substitutes

- Often additive independence is not a good assumption.
- Values x_1 of feature X_1 and x_2 of feature X_2 are complements if having both is better than the sum of the two.
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  - A trip to a location 3 hours North on day 3
  - The return trip for the same day.

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- x_i and x_j are substitutes iff

Canonical Representation of Utility

- Generalized additive utility defines utility in terms of sum of factors, where a factor is a function on some of the variables.
- The canonical representation of utility allows weights for conjunctions of feature values. For Boolean $\{0, 1\}$ features:

$$\begin{aligned} u(x_1, \ldots, x_n) = & \ w_0 + w_1 * x_1 + w_2 * x_2 \cdots + w_n * x_n \\ & + w_{12} * x_1 * x_2 + w_{13} * x_1 * x_3 + \ldots \\ & + w_{123} * x_1 * x_2 * x_3 + \ldots \\ & \ldots \end{aligned}$$

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Utility and time

- Would you prefer \$1000 today or \$1000 next year?
- What price would you pay now to have an eternity of happiness?
- How can you trade off pleasures today with pleasures in the future?

Pascal's Wager (1670)

Decide whether to believe in God.

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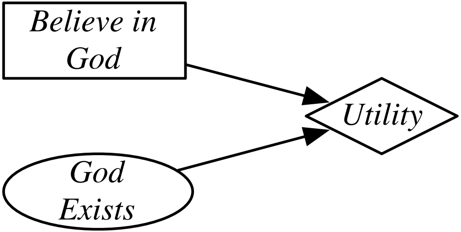

Utility and time

- How would you compare the following sequences of rewards (per week):

  A: \$1000000, \$0, \$0, \$0, \$0, \$0,...
  B: \$1000, \$1000, \$1000, \$1000, \$1000,...
  C: \$1000, \$0, \$0, \$0, \$0,...
  D: \$1, \$1, \$1, \$1, \$1,...
  E: \$1, \$2, \$3, \$4, \$5,...

Rewards and Values

Suppose the agent receives a sequence of rewards $r_1, r_2, r_3, r_4, \ldots$ in time. What utility should be assigned? "Return" or "value"

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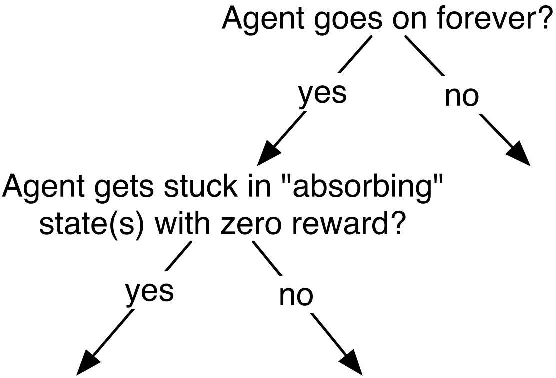
- total reward $V = \sum_{i=1}^{\infty} r_i$
- average reward $V = \lim_{n\to\infty} (r_1 + \cdots + r_n)/n$

Average vs Accumulated Rewards



Rewards and Values

Suppose the agent receives a sequence of rewards $r_1, r_2, r_3, r_4, \ldots$ in time.

- discounted return $V = r_1 + \gamma r_2 + \gamma^2 r_3 + \gamma^3 r_4 + \cdots$
  γ is the discount factor $0 \leq \gamma \leq 1$.

Properties of the Discounted Rewards

- The discounted return for rewards $r_1, r_2, r_3, r_4, \ldots$ is

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Therefore $\dfrac{\text{minimum reward}}{1-\gamma} \leq V_t \leq \dfrac{\text{maximum reward}}{1-\gamma}$
- You can approximate V with the first k terms, with error:
$$\begin{aligned} V - (r_1 + \gamma r_2 + \cdots + \gamma^{k-1} r_k) &= \gamma^k V_{k+1} \\ &\propto \gamma^k/(1-\gamma) \end{aligned}$$

Properties of the Discounted Rewards

- $V = r_1 + \gamma r_2 + \gamma^2 r_3 + \gamma^3 r_4 + \cdots$
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  - with probability γ, agent keeps going
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- γ should reflect an agent's utility.

Why discounting? [Koopmans 1972]

With an infinite sequence of outcomes $\langle o_1, o_2, o_3, \ldots \rangle$ if

- the first time period matters, so $\exists\ o_1, o_2, o_3, \ldots$ and o_1' where

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- the agent only cares about finite subspaces of infinite time

then there exists a discount factor γ and function r such that

$$utility(\langle o_1, o_2, o_3, \ldots \rangle) = \sum_i \gamma^{i-1} r(o_i)$$

Allais Paradox (1953)

What would you prefer:

A: $\$1m$ — one million dollars

B: lottery $[0.10 : \$2.5m, 0.89 : \$1m, 0.01 : \$0]$

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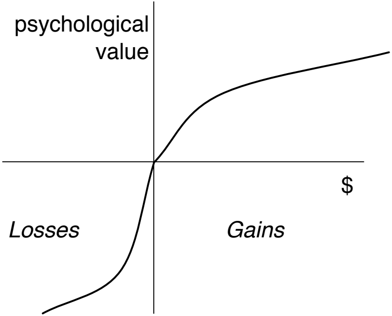
A,C: lottery $[0.11 : \$1m, 0.89 : X]$

B,D: lottery $[0.10 : \$2.5m, 0.01 : \$0, 0.89 : X]$

Prospect Theory



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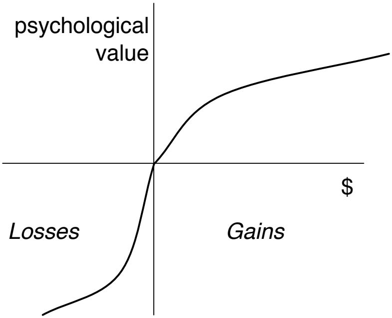

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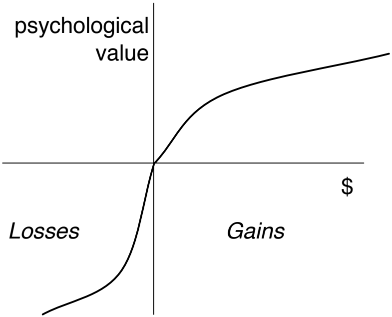

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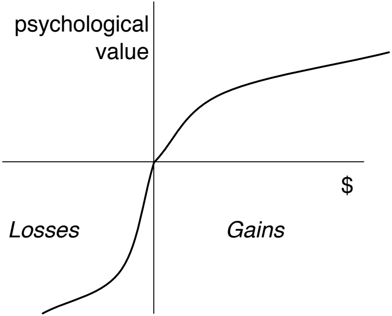

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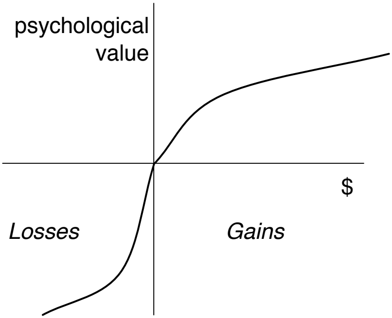

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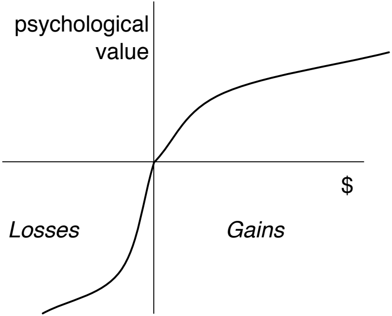

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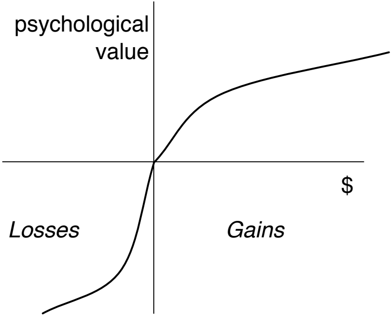

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This better fits with human preferences.

Reference Points

Consider Anthony and Betty who (for argument) are essentially the same except:

- Anthony's current wealth is \$1 million.
- Betty's current wealth is \$4 million.

They are both offered the choice between a gamble and a sure thing:

- Gamble: equal chance to end up owning \$1 million or \$4 million.
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What does prospect theory predict?
Anthony is making a gain so will will be risk averse and take the sure thing.
Better is making a loss and so will will be risk seeking and gamble.

Reference Points [Kahneman 2011]

Twins Andy and Bobbie, have identical tastes and identical starting jobs. There are two jobs that are identical, except that

- job A gives a raise of \$10000
- job B gives an extra day of vacation per month.

They are each indifferent to the outcomes and toss a coin. Andy takes job A, and Bobbie takes job B.
Now the company suggests they swap jobs with a \$500 bonus.
Will they swap?

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Utility theory predicts they swap. Prospect theory predicts they do not swap.
[From D. Kahneman, *Thinking, Fast and Slow*, 2011, p. 291.]

Framing Effects [Tversky and Kahneman]

- A disease is expected to kill 600 people. Two alternative programs have been proposed:

  Program A: 200 people will be saved
  Program B: probability $1/3$: 600 people will be saved
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  Which program would you favor?

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- A disease is expected to kill 600 people. Two alternative programs have been proposed:

  Program C: 400 people will die
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Tversky and Kahneman: 72% chose A over B.
22% chose C over D.

Framing Effects

What do you think of Alan and Ben:

- Alan: intelligent—industrious—impulsive—critical—stubborn—envious

Framing Effects

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[From D. Kahneman, Thinking Fast and Slow, 2011, p. 82]

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[From R.M. Dawes, Rational Choice in an Uncertain World, 1988.]

The Ellsberg Paradox

Two bags:

- **Bag 1** 40 white chips, 30 yellow chips, 30 green chips
- **Bag 2** 40 white chips, 60 chips that are yellow or green

What do you prefer:

- A: Receive \$1m if a white or yellow chip is drawn from bag 1
- B: Receive \$1m if a white or yellow chip is drawn from bag 2
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However A and D should give same outcome, no matter what the proportion in Bag 2.

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- Is it rational to gamble o_1 to on a coin toss to get o_2?
- Is it rational to gamble o_2 to on a coin toss to get o_3?
- Is it rational to gamble o_3 to on a coin toss to get o_4?

St. Petersburg Paradox

What if there is no "best" outcome?
Are utilities unbounded?

- Suppose utilities are unbounded.
- Then for any outcome o_i there is an outcome o_{i+1} such that $u(o_{i+1}) > 2u(o_i)$.
- Would the agent prefer o_1 or the lottery $[0.5 : o_2, 0.5 : 0]$ where 0 is the worst outcome?
- Is it rational to gamble o_1 to on a coin toss to get o_2?
- Is it rational to gamble o_2 to on a coin toss to get o_3?
- Is it rational to gamble o_3 to on a coin toss to get o_4?
- What will eventually happen?

Predictor Paradox

Two boxes:

Box 1: contains $10,000

Box 2: contains either $0 or $1m

- You can either choose both boxes or just box 2.

Predictor Paradox

Two boxes:

Box 1: contains \$10,000

Box 2: contains either \$0 or \$1m

- You can either choose both boxes or just box 2.
- The "predictor" has put \$1m in box 2 if he thinks you will take box 2 and \$0 in box 2 if he thinks you will take both.
- The predictor has been correct in previous predictions.
- Do you take both boxes or just box 2?