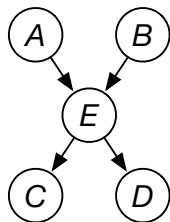


Learning a Belief Network

- If you
 - ▶ know the structure
 - ▶ have observed all of the variables
 - ▶ have no missing data
 - you can learn each conditional probability separately.
- supervised learning

Learning belief network example

Model



Data

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>t</i>	<i>f</i>	<i>t</i>	<i>t</i>	<i>f</i>
<i>f</i>	<i>t</i>	<i>t</i>	<i>t</i>	<i>t</i>
<i>t</i>	<i>t</i>	<i>f</i>	<i>t</i>	<i>f</i>
		...		

→ Probabilities

$P(A)$
 $P(B)$
 $P(E | A, B)$
 $P(C | E)$
 $P(D | E)$

Learning conditional probabilities

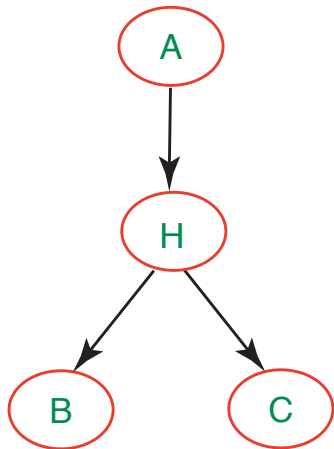
- Each conditional probability distribution can be learned separately:
- For example:

$$P(E = t \mid A = t \wedge B = f) \\ = \frac{(\# \text{examples: } E = t \wedge A = t \wedge B = f) + c_1}{(\# \text{examples: } A = t \wedge B = f) + c}$$

where c_1 and c reflect prior (expert) knowledge ($c_1 \leq c$).

- When there are many parents to a node, there can be little or no data for each conditional probability: use supervised learning to learn a decision tree, linear classifier, a neural network or other representation of the conditional probability.

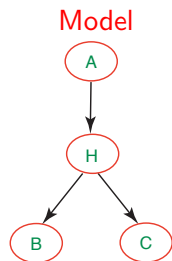
Unobserved Variables



- What if you had only observed values for A, B, C?

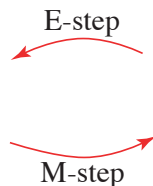
A	B	C
<i>t</i>	<i>f</i>	<i>t</i>
<i>f</i>	<i>t</i>	<i>t</i>
<i>t</i>	<i>t</i>	<i>f</i>
...		

EM Algorithm



Augmented Data

<i>A</i>	<i>B</i>	<i>C</i>	<i>H</i>	<i>Count</i>
<i>t</i>	<i>f</i>	<i>t</i>	<i>t</i>	0.7
<i>t</i>	<i>f</i>	<i>t</i>	<i>f</i>	0.3
<i>f</i>	<i>t</i>	<i>t</i>	<i>f</i>	0.9
<i>f</i>	<i>t</i>	<i>t</i>	<i>t</i>	0.1



Probabilities

$$P(A)$$
$$P(H | A)$$
$$P(B | H)$$
$$P(C | H)$$

- Repeat the following two steps:
 - ▶ **E-step** give the expected number of data points for the unobserved variables based on the given probability distribution. Requires probabilistic inference.
 - ▶ **M-step** infer the (maximum likelihood) probabilities from the data. This is the same as the fully-observable case.
- Start either with made-up data or made-up probabilities.
- EM will converge to a local maxima.

Belief network structure learning (I)

Given examples \mathbf{e} , and model m :

$$P(m | \mathbf{e}) = \frac{P(\mathbf{e} | m) * P(m)}{P(\mathbf{e})}.$$

- A model here is a belief network.
 - A bigger network can always fit the data better.
 - $P(m)$ lets us encode a preference for simpler models (e.g, smaller networks)
- search over network structure looking for the most likely model.
- Taking logarithms (and negating):

$$\arg \max_m P(m | \mathbf{e}) = \arg \min_m (-\log P(\mathbf{e} | m) - \log P(m))$$

Bayes Rule:

$$P(h|d) \propto P(d|h)P(h)$$

$$\begin{aligned}\arg \max_h P(h|d) &= \arg \max_h P(d|h)P(h) \\ &= \arg \max_h (\log P(d|h) + \log P(h))\end{aligned}$$

- $\log P(d|h)$ measures fit to data
- $\log P(h)$ measures model complexity

Information theory overview

- A **bit** is a binary digit.
- 1 bit can distinguish 2 items
- k bits can distinguish 2^k items
- n items can be distinguished using $\log_2 n$ bits
- Can you do better?

Consider a code to distinguish elements of $\{a, b, c, d\}$ with

$$P(a) = \frac{1}{2}, P(b) = \frac{1}{4}, P(c) = \frac{1}{8}, P(d) = \frac{1}{8}$$

Consider the code:

a 0 b 10 c 110 d 111

This code uses 1 to 3 bits. On average, it uses

$$\begin{aligned} &P(a) * 1 + P(b) * 2 + P(c) * 3 + P(d) * 3 \\ &= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{3}{8} = 1\frac{3}{4} \text{ bits.} \end{aligned}$$

The string *aacabbda* has code 00110010101110.

The code 0111110010100 represents string *adcabba*

- To identify x , you need $-\log_2 P(x)$ bits.
- Given a distribution over a set of values, to identify a member, the expected number of bits is

$$\sum_x -P(x) * \log_2 P(x)$$

is the **information content** or **entropy** of the distribution.

- The expected number of bits it takes to describe a distribution given evidence e :

$$I(e) = \sum_x -P(x|e) * \log_2 P(x|e).$$

Given a test that can distinguish the cases where α is true from the cases where α is false, the **information gain** from this test is:

$$I(\text{true}) - (P(\alpha) * I(\alpha) + P(\neg\alpha) * I(\neg\alpha)).$$

- $I(\text{true})$ is the expected number of bits needed before the test
- $P(\alpha) * I(\alpha) + P(\neg\alpha) * I(\neg\alpha)$ is the expected number of bits after the test.

Comparing Distributions

- Suppose a code is designed to be optimal for probability distribution Q , so that x is described using $-\log_2 Q(x)$ bits.
- Suppose P is another probability distribution. The expected number of bits to describe P using the code for Q is

$$\sum_x -P(x) * \log_2 Q(x)$$

- The difference between this and the entropy of P — describing P using its optimal code — is the **Kullback–Leibler (KL) divergence** (also called **relative entropy**):

$$\begin{aligned} D_{KL}(P \parallel Q) &= \left(\sum_x -P(x) * \log Q(x) \right) - \sum_x -P(x) * \log P(x) \\ &= \sum_x -P(x) * \log(Q(x)/P(x)) \end{aligned}$$

- When is this large? When $P(x) \gg Q(x)$ for some x .

A belief network structure learning algorithm

- Search over total orderings of variables.
- For each total ordering X_1, \dots, X_n use supervised learning to learn $P(X_i | X_1 \dots X_{i-1})$.
- Return the network model found with minimum:
 - $\log P(\mathbf{e} | m) - \log P(m)$
 - ▶ $P(\mathbf{e} | m)$ can be obtained by inference.
 - ▶ How to determine $-\log P(m)$?

Bayesian Information Criterion (BIC) Score

$$P(m | \mathbf{e}) = \frac{P(\mathbf{e} | m) * P(m)}{P(\mathbf{e})}$$

$$-\log P(m | \mathbf{e}) \propto -\log P(\mathbf{e} | m) - \log P(m)$$

- $-\log P(\mathbf{e} | m)$ is the negative log likelihood of model m : number of bits to describe the data in terms of the model.
- $|\mathbf{e}|$ is the number of examples. Each proposition can be true for between 0 and $|\mathbf{e}|$ examples, so there are $|\mathbf{e}| + 1$ different probabilities to distinguish. Each one can be described in $\log(|\mathbf{e}| + 1) \approx \log(|\mathbf{e}|)$ bits.
- If there are $\|m\|$ independent parameters ($\|m\|$ is the dimensionality of the model):

$$-\log P(m | \mathbf{e}) \propto -\log P(\mathbf{e} | m) + \|m\| \log(|\mathbf{e}|)$$

This is the **Bayesian Information Criterion (BIC)** score.

Belief network structure learning (II)

- Given a total ordering, to determine $parents(X_i)$ do independence tests to determine which features should be the parents
- XOR problem: just because features do not give information individually, does not mean they will not give information in combination
- Search over total orderings of variables

- You cannot just ignore missing data unless you know it is missing at random.
- Is the reason data is missing correlated with something of interest?
- For example: data in a clinical trial to test a drug may be missing because:
 - ▶ the patient dies
 - ▶ the patient had severe side effects
 - ▶ the patient was cured
 - ▶ the patient had to visit a sick relative.

— ignoring some of these may make the drug look better or worse than it is.
- In general you need to model why data is missing (see Chapter 11)

General Learning of Belief Networks

- We have a mixture of observational data and data from randomized studies.
- We are not given the structure.
- We don't know whether there are hidden variables or not. We don't know the domain size of hidden variables.
- There is missing data.

... this is too difficult for current techniques!