Variable Elimination

- Variable elimination is the dynamic programming variant of recursive conditioning.
- Give a factorization, such as

$$P(D) = \sum_C P(D \mid C) \sum_B P(C \mid B) \sum_A P(A)P(B \mid A)$$

  it does the innermost sums first, constructing representations of the intermediate factors:
  - $\sum_A P(A)P(B \mid A)$ is a factor on B; call it $f_1(B)$.
  - $\sum_B P(C \mid B) f_1(B)$ is a factor on C.
- Lecture covers:
  - Factors and factor arithmetic
  - Variable elimination algorithm

Factors

- A factor is a representation of a function from a tuple of random variables into a number.
- We write factor f on variables $X_1, \ldots, X_j$ as $f(X_1, \ldots, X_j)$.
- You can assign some or all of the variables of a factor:
  - $f(X_1 = v_1, X_2, \ldots, X_j)$, where $v_1 \in domain(X_1)$, is a factor on $X_2, \ldots, X_j$.
  - $f(X_1 = v_1, X_2 = v_2, \ldots, X_j = v_j)$ is a number that is the value of f when each X_i has value v_i.

The former is also written as $f(X_1, X_2, \ldots, X_j)_{X_1 = v_1}$, etc.

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$r(X{=}t, Y, Z{=}f)$:

Y	val
t	0.9
f	0.8

$r(X{=}t, Y{=}f, Z{=}f) = 0.8$

Multiplying factors

The product of factor $f_1(\overline{X}, \overline{Y})$ and $f_2(\overline{Y}, \overline{Z})$, where $\overline{Y}$ are the variables in common, is the factor $(f_1 * f_2)(\overline{X}, \overline{Y}, \overline{Z})$ defined by:

$$(f_1 * f_2)(\overline{X}, \overline{Y}, \overline{Z}) \quad = \quad f_1(\overline{X}, \overline{Y}) f_2(\overline{Y}, \overline{Z}).$$

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

Summing out variables

We can sum out a variable, say X_1 with domain $\{v_1, \ldots, v_k\}$, from factor $f(X_1, \ldots, X_j)$, resulting in a factor on $X_2, \ldots, X_j$ defined by:

$$\begin{aligned}&(\sum_{X_1} f)(X_2, \ldots, X_j)\\ =\ & f(X_1 = v_1, \ldots, X_j) + \cdots + f(X_1 = v_k, \ldots, X_j)\end{aligned}$$

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	0.43
f	t	0.54
f	f	0.46

Queries and Evidence

- To compute the posterior probability of query Q given evidence $E=e$:

$$
\begin{aligned}
& P(Q \mid E=e) \\
&\quad = \frac{P(Q, E=e)}{P(E=e)} \\
&\quad = \frac{P(Q, E=e)}{\sum_Q P(Q, E=e)}.
\end{aligned}
$$

- So the computation reduces to the probability of $P(Q, E=e)$
- then normalize at the end.

Probability of a conjunction

- The variables of the belief network are $X_1, \ldots, X_n$.
- The evidence is $Y_1 = v_1, \ldots, Y_j = v_j$
- To compute $P(Q, Y_1 = v_1, \ldots, Y_j = v_j)$:
  we add the other variables,
  $Z_1, \ldots, Z_k = \{X_1, \ldots, X_n\} - \{Q\} - \{Y_1, \ldots, Y_j\}$.
  and sum them out.
- We order the Z_i into an elimination ordering.

$$
\begin{aligned}
& P(Q, Y_1 = v_1, \ldots, Y_j = v_j) \\
& \quad = \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \ldots, X_n)_{Y_1 = v_1, \ldots, Y_j = v_j}. \\
& \quad = \sum_{Z_k} \cdots \sum_{Z_1} \prod_{i=1}^{n} P(X_i \mid parents(X_i))_{Y_1 = v_1, \ldots, Y_j = v_j}.
\end{aligned}
$$

Computing sums of products

Computation in belief networks reduces to computing the sums of products.

- How can we compute $ab + ac$ efficiently?
- Distribute out a giving $a(b + c)$
- How can we compute $\sum_{Z_1} \prod_{i=1}^{n} P(X_i \mid parents(X_i))$ efficiently?
- Distribute out those factors that don't involve Z_1.

Variable elimination algorithm

To compute $P(Q \mid Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j)$:

- Construct a factor for each conditional probability.
- Set the observed variables to their observed values.
- Sum out each of the non-observed non-query variables (the $\{Z_1, \ldots, Z_k\}$) according to some elimination ordering.
- Multiply the remaining factors.
- Normalize by dividing the resulting factor $f(Q)$ by $\sum_Q f(Q)$.

Summing out a variable

To sum out a variable Z_j from a product $f_1, \ldots, f_k$ of factors:

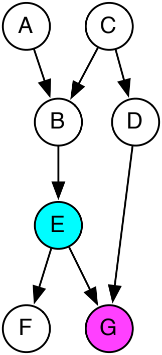
- Partition the factors into
  - those that don't contain Z_j, say $f_1, \ldots, f_i$,
  - those that contain Z_j, say $f_{i+1}, \ldots, f_k$

Then:

$$\sum_{Z_j} f_1 * \cdots * f_k = f_1 * \cdots * f_i * \left(\sum_{Z_j} f_{i+1} * \cdots * f_k \right).$$

- Explicitly construct a representation of the rightmost factor. Replace the factors $f_{i+1}, \ldots, f_k$ by the new factor.

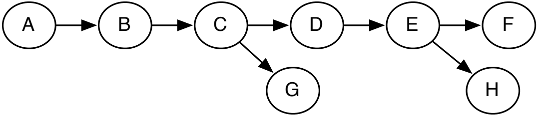
Example



$$P(E \mid g) = \frac{P(E \wedge g)}{\sum_E P(E \wedge g)}$$

$$\begin{aligned}
& P(E \wedge g) \\
&= \sum_F \sum_B \sum_C \sum_A \sum_D P(A)P(B \mid AC) \\
&\quad P(C)P(D \mid C)P(E \mid B)P(F \mid E)P(g \mid ED) \\
&= \left(\sum_F P(F \mid E)\right) \\
&\quad \sum_B P(E \mid B) \sum_C \left(P(C) \left(\sum_A P(A)P(B \mid AC)\right)\right. \\
&\quad \left.\left(\sum_D P(D \mid C)P(g \mid ED)\right)\right)
\end{aligned}$$

Variable Elimination example



Query: $P(G \mid f)$; elimination ordering: A, H, E, D, B, C

$$P(G \mid f) \;\propto\; \sum_C \sum_B \sum_D \sum_E \sum_H \sum_A P(A)P(B \mid A)P(C \mid B)$$
$$P(D \mid C)P(E \mid D)P(f \mid E)P(G \mid C)P(H \mid E)$$

$$= \sum_C \left(\sum_B \left(\sum_A P(A)P(B \mid A) \right) P(C \mid B) \right) P(G \mid C)$$
$$\left(\sum_D P(D \mid C) \left(\sum_E P(E \mid D)P(f \mid E) \sum_H P(H \mid E) \right) \right)$$

Pruning Irrelevant Variables (Belief networks)

Suppose you want to compute $P(X \mid e_1 \ldots e_k)$:

- Prune any variables that have no observed or queried descendents.
- Connect the parents of any observed variable.
- Remove arc directions.
- Remove observed variables.
- Remove any variables not connected to X in the resulting (undirected) graph.

Variable Elimination and Recursive Conditioning

- Variable elimination is the dynamic programming variant of recursive conditioning.
- Recursive Conditioning is the search variant of variable elimination.
- They do the same additions and multiplications.
- Space and time complexity $O(nd^t)$, for n variables, domain size d, and treewidth t.
  – treewidth is the number of variables in the smallest factor.
  It is a property of the graph and the elimination ordering.
- Recursive conditioning never modifies or creates factors; it only evaluates them.