Understanding Independence: Common ancestors

- *Alarm* and *Smoke* are dependent given {}.
- *Alarm* and *Smoke* are independent given *Fire*.
- Intuitively, *Fire* can explain *Alarm* and *Smoke*; learning one can affect the other by changing the belief in *Fire*.
Alarm and Report are dependent given \{\}\n
Alarm and Report are independent given Leaving

The (only) way that the Alarm affects Report is by affecting Leaving.
Tampering and Fire are independent given $\emptyset$

Tampering and Fire are dependent given Alarm

Intuitively, Tampering can explain away Fire
(a) On which given probabilities does $P(N)$ depend?

(b) If you were to observe a value for $B$, which variables’ probabilities will change?

(c) If you were to observe a value for $N$, which variables’ probabilities will change?
(d) Suppose you had observed a value for \( M \); if you were to then observe a value for \( N \), which variables’ probabilities will change?

(e) Suppose you had observed \( B \) and \( Q \); which variables’ probabilities will change when you observe \( N \)?
If you observe variable(s) \( \overline{Y} \), the variables whose posterior probability is different from their prior are:

- ancestors of \( \overline{Y} \) and
- their descendants.

Intuitively (assuming network ordered so causes are before effects):

- You do **abduction** to possible causes and
- **prediction** from the causes.
Three types of meetings between arcs

(a) chain  (b) fork  (c) collider
D-separation

- A **path** $p$ can follow arrows in either direction.
- Observations $Z$s **block** a path $p$ if:
  1. $p$ contains a **chain** $A \rightarrow B \rightarrow C$, and $B \in Z$s
  2. $p$ contains a **fork** $A \leftarrow B \rightarrow C$, and $B \in Z$s
  3. $p$ contains a **collider** $A \rightarrow B \leftarrow C$, and $B$, and all its descendants, are **not** in $Z$s
- $X$ is **d-separated** from $Y$ given $Z$s if every path between $X$ and $Y$ is blocked by $Z$s
- $X$ is independent $Y$ given $Z$s for all conditional probabilities iff $X$ is d-separated from $Y$ given $Z$s
Example

- Are $X$ and $Y$ d-separated by $\emptyset$?
- Are $X$ and $Y$ d-separated by $\{K\}$?
- Are $X$ and $Y$ d-separated by $\{K, N\}$?
- Are $X$ and $Y$ d-separated by $\{K, N, P\}$?
A Markov random field is composed of

- of a set of random variables: $X = \{X_1, X_2, \ldots, X_n\}$ and
- a set of factors $\{f_1, \ldots, f_m\}$, where a factor is a non-negative function of a subset of the variables.

and defines a joint probability distribution:

$$P(X = x) \propto \prod_{k} f_k(X_k = x_k).$$

$$P(X = x) = \frac{1}{Z} \prod_{k} f_k(X_k = x_k).$$

$$Z = \sum_{x} \prod_{k} f_k(X_k = x_k)$$

where $f_k(X_k)$ is a factor on $X_k \subseteq X$, and $x_k$ is $x$ projected onto $X_k$. $Z$ is a normalization constant known as the partition function.
A **factor graph** is a bipartite graph, which contains a variable node for each random variable and a factor node for each factor. There is an edge between a variable node and a factor node if the variable appears in the factor.

A **Markov network** is a graphical representation of a Markov random field where the nodes are the random variables and there is an arc between any two variables that are in a factor together.

A **belief network** is a type of Markov random field where the factors represent conditional probabilities, there is a factor for each variable, and directed graph is acyclic.
Factor graph and Markov network example

Factor Graph

Markov Network

\[ f(A, B, C) \]
\[ f(B, D) \]
\[ f(C, E) \]

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The Markov blanket of a variable $X$ is the set of variables that are in factors with $X$.

A variable is independent of the other variables given its Markov blanket.

$X$ is connected to $Y$ given $\overline{Z}$ if there is a path from $X$ to $Y$ in the Markov network, which does not contain an element of $Z$.

$X$ is separated from $Y$ given $\overline{Z}$ if it is not connected.

A positive factor is one that does not contain zero values.

$\overline{X}$ is independent $\overline{Y}$ given $\overline{Z}$ for all positive factors iff $\overline{X}$ is separated from $\overline{Y}$ given $\overline{Z}$.
The parameters of a graphical model are the numbers that define the model.

A belief network is a canonical representation: given the structure and the distribution, the parameters are uniquely determined.

A Markov random field is not a canonical representation. Many different parameterizations result in the same distribution.