Conditional independence

Random variable X is independent of random variable Y given random variable(s) Z if,

$$P(X \mid Y, Z) = P(X \mid Z)$$

i.e. for all $x_i \in domain(X)$, $y_j \in domain(Y)$, $y_k \in domain(Y)$ and $z_m \in domain(Z)$,

$$\begin{aligned} & P(X = x_i \mid Y = y_j \wedge Z = z_m) \\ = \;& P(X = x_i \mid Y = y_k \wedge Z = z_m) \\ = \;& P(X = x_i \mid Z = z_m). \end{aligned}$$

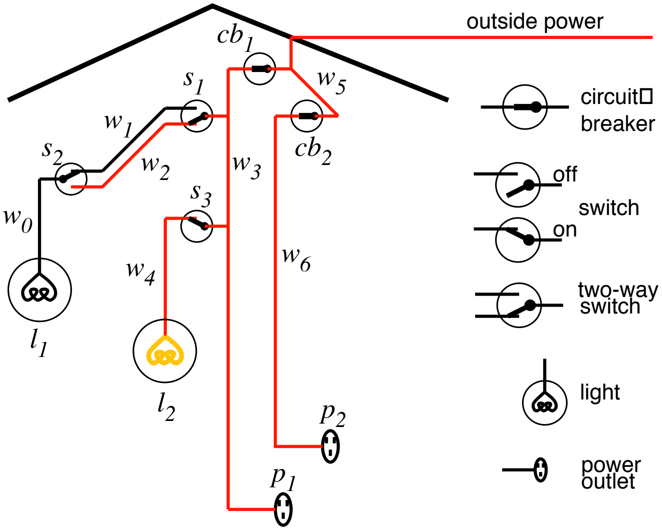
That is, knowledge of Y's value doesn't affect the belief in the value of X, given a value of Z.

Example

Consider a student writing an exam.
What are reasonable independences among the following?

- Whether the student works hard (W)
- Whether the student is intelligent (I)
- The student's answers on the exam (A)
- The student's mark on an exam (M)

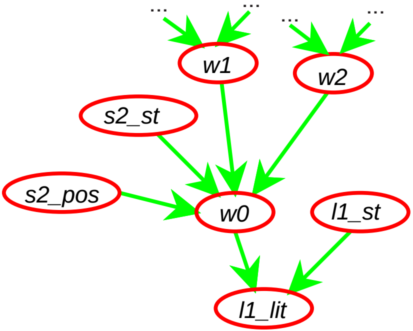
Example domain (diagnostic assistant)



Examples of conditional independence?

- The party forming govenmnent of Canada is dependent or independent of whether light l_1 is lit given whether there is outside power?
- Whether there is someone in a room is independent of whether a light l_2 is lit given the position of switch $s3$.
- Whether light l_1 is lit is independent of the position of light switch s_2 given whether there is power in wire w_0.
- Every other variable may be independent of whether light l_1 is lit given whether there is power in wire w_0 and the status of light l_1 (if it's ok, or if not, how it's broken).

- l_1 is lit ($L1\_lit$) depends only on the status of the light ($L1\_st$) and whether there is power in wire $w0$.
- In a belief network, $W0$ and $L1\_st$ are parents of $L1\_lit$.



- $W0$ depends only on whether there is power in $w1$, whether there is power in $w2$, the position of switch $s2$ ($S2\_pos$), and the status of switch $s2$ ($S2\_st$).

Belief networks

- Totally order the variables of interest: $X_1, \ldots, X_n$
- Theorem of probability theory (chain rule): $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i \mid X_1, \ldots, X_{i-1})$
- The parents of X_i, $parents(X_i)$, are those predecessors of X_i that render X_i independent of the other predecessors. That is,

$$parents(X_i) \subseteq X_1, \ldots, X_{i-1}$$

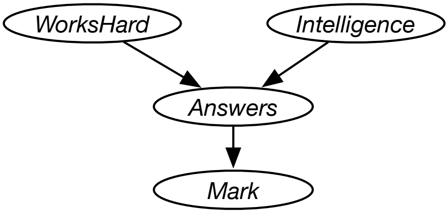
$$P(X_i \mid parents(X_i)) = P(X_i \mid X_1, \ldots, X_{i-1})$$

- So $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i \mid parents(X_i))$
- A belief network is a graph: the nodes are random variables; there is an arc from the parents of each node into that node.

Student Writing an Exam Example

Give a belief network for the variables in order:

- *WorksHard*: Whether the student works hard
- *Intelligent*: Whether the student is intelligent
- *Answers*: The student's answers on the exam
- *Mark*: The student's mark on an exam

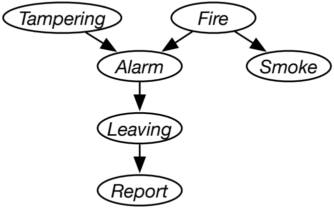


What if the variables were in the opposite order?

Example: fire alarm belief network

Variables:

- Fire: there is a fire in the building
- Tampering: someone has been tampering with the fire alarm
- Smoke: what appears to be smoke is coming from a window
- Alarm: the fire alarm goes off
- Leaving: people are leaving the building *en masse*.
- Report: told people are leaving the building *en masse*.



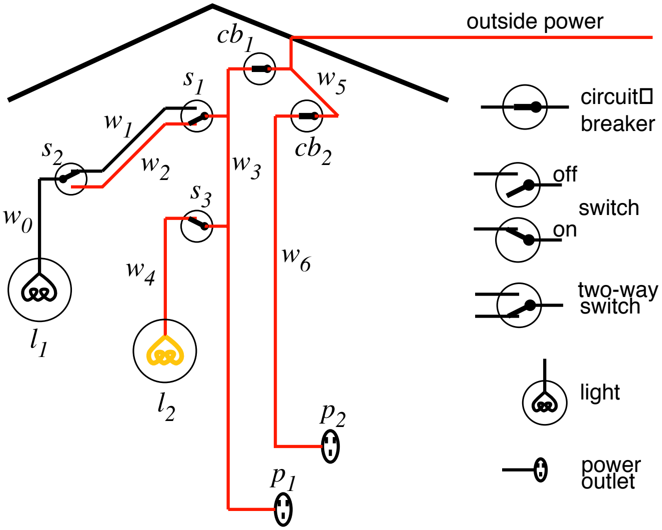
AIPython.org: `bn_report` in `probGraphicalModels.py`

Components of a belief network

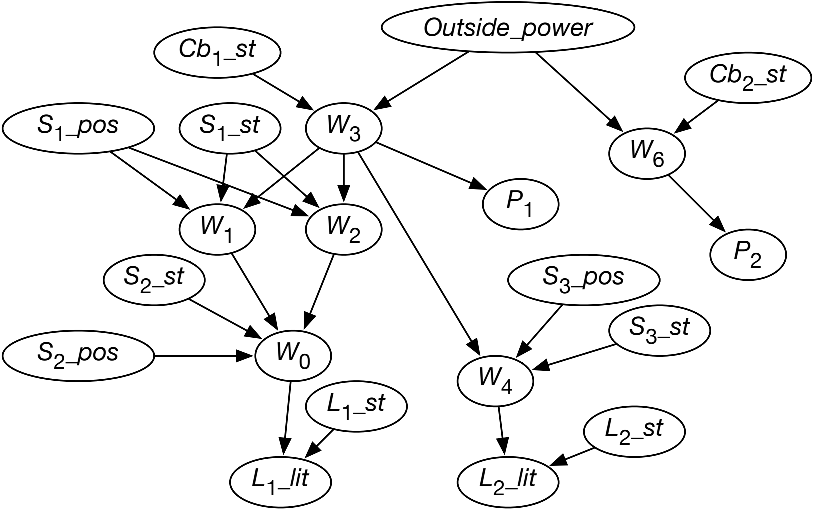
A belief network consists of:

- a directed acyclic graph with nodes labeled with random variables
- a domain for each random variable
- a set of conditional probabilities, one for each variable given its parents (including prior probabilities for nodes with no parents).

Task Domain: Electrical Environment



Example belief network



Example belief network (continued)

The belief network also specifies:

- The domain of the variables:
  $W_0, \ldots, W_6$ have domain $\{live, dead\}$
  S_1_pos, S_2_pos, and S_3_pos have domain $\{up, down\}$
  S_1_st has $\{ok, upside\_down, short, intermittent, broken\}$.
- Conditional probabilities, including:
  $P(W_1 = live \mid s_1\_pos = up \land S_1\_st = ok \land W_3 = live)$
  $P(W_1 = live \mid s_1\_pos = up \land S_1\_st = ok \land W_3 = dead)$
  $P(S_1\_pos = up)$
  $P(S_1\_st = upside\_down)$

Belief network summary

- A belief network is a directed acyclic graph (DAG) where nodes are random variables.
- The parents of a node N are those variables on which N directly depends.
- A belief network is automatically acyclic by construction.
- A belief network is a graphical representation of dependence and independence:
  - A variable is independent of its non-descendants given its parents.

Constructing belief networks

To represent a domain in a belief network, you need to consider:

- What are the relevant variables?
  - What will you observe?
  - What would you like to find out (query)?
  - What other features make the model simpler?
- What values should these variables take?
- What is the relationship between the variables?
  This is in terms of a directed graph, representing conditional dependence.
- How does the value of each variable depend on its parents?
  This is expressed in terms of the conditional probabilities.