Learning Objectives

At the end of the class you should be able to:

- devise an useful heuristic function for a problem
- demonstrate how best-first and \( A^* \) search will work on a graph
- predict the space and time requirements for best-first and \( A^* \) search
Heuristic Search

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- $h(n)$ is an **underestimate** if there is no path from $n$ to a goal with cost less than $h(n)$.
- An **admissible heuristic** is a heuristic function that is an underestimate of the actual cost of a path to a goal.
Example Heuristic Functions

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- A heuristic function can be found by solving a simpler (less constrained) version of the problem.
Idea: in depth-first search select a neighbor that is closest to a goal according to the heuristic function.
Heuristic depth-first Search

- **Idea:** in depth-first search select a neighbor that is closest to a goal according to the heuristic function.
- It inherits all of the advantages/disadvantages of depth-first search, but locally heads towards a goal.
Best-first Search

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- Best-first search selects a path on the frontier with minimal $h$-value.
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- Best-first search selects a path on the frontier with minimal $h$-value.
- It treats the frontier as a priority queue ordered by $h$. 
Illustrative Graph — Heuristic Search
Does best-first search guarantee to find a least-cost path?
Complexity of Best-first Search

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- Does best-first search guarantee to find a path with fewest arcs?
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- How does the goal affect the search?
A* Search

- A* search uses both path cost and heuristic values.

\[ f(p) = \text{cost}(p) + h(p) \]

In A* search, the frontier is a priority queue ordered by \( f(p) \). It always selects the path on the frontier with the lowest estimated cost from the start to a goal node constrained to go via that path.
A* Search

- A* search uses both path cost and heuristic values
- \( cost(p) \) is the cost of path \( p \).
- \( h(p) \) estimates the cost from the end of \( p \) to a goal.
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- \( \text{cost}(p) \) is the cost of path \( p \).
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- Let \( f(p) = \text{cost}(p) + h(p) \).
  
  \( f(p) \) estimates the total path cost of going from a start node to a goal via \( p \).

![Diagram](https://via.placeholder.com/150)

\[ \begin{align*}
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Admissibility of $A^*$

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- the branching factor is finite
- arc costs are bounded above zero (there is some $\epsilon > 0$ such that all of the arc costs are greater than $\epsilon$), and
- $h(n)$ is nonnegative and an underestimate of the cost of the shortest path from $n$ to a goal node:

$$0 \leq h(n) \leq \text{cost of shortest path from } n \text{ to a goal}$$
Why is $A^*$ admissible?

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- If a path $p$ to a goal is selected from the frontier, can there be a lower cost path to a goal?
- $h(p) = 0$
- Suppose path $p'$ is on the frontier. Because $p$ was chosen before $p'$, and $h(p) = 0$: 
  
  $$\text{cost}(p) \leq \text{cost}(p') + h(p')$$
  
  Because $h$ is an underestimate:
  
  $$\text{cost}(p') + h(p') \leq \text{cost}(p'')$$
  
  for any path $p''$ to a goal that extends $p'$. So $\text{cost}(p) \leq \text{cost}(p'')$ for any other path $p''$ to a goal.
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- The frontier always contains the initial part of a path to a goal, before that goal is selected.
- $A^*$ halts, as the costs of the paths on the frontier keeps increasing, and will eventually exceed any finite number.
How do good heuristics help?

Suppose $c$ is the cost of an optimal solution. What happens to a path $p$ from a start node, where

- $\text{cost}(p) + h(p) < c$

It will be expanded.

- $\text{cost}(p) + h(p) > c$

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- $\text{cost}(p) + h(p) = c$

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<th>Strategy</th>
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<th>Complete</th>
<th>Halts</th>
<th>Space</th>
</tr>
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<tbody>
<tr>
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<td></td>
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### Summary of Search Strategies

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