Searching Possible Worlds

- Can we estimate the probabilities by only enumerating a few of the possible worlds?
- How can we enumerate just a few of the most probable possible worlds?
- Can we estimate the error in our estimates?
- Can we exploit the structure that variable elimination does?
- Can we exploit more structure?

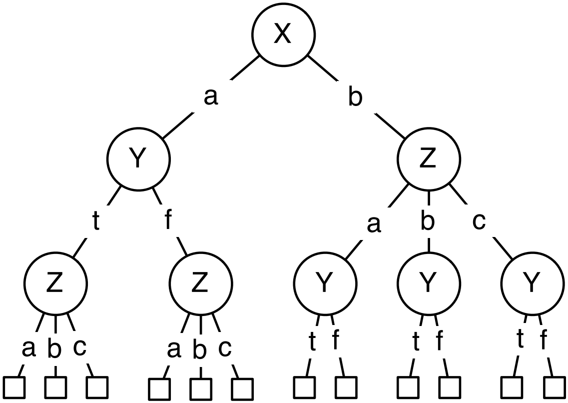
Search tree

The search tree has nodes labeled with variables, and is defined as follows:

- Each non-leaf node is labelled with a variable
- The arcs are labelled with values. There is a child for a node X for every value in the domain of X.
- A node cannot be labelled with the same label as an ancestor node.
- A path from the root corresponds to an assignment to a set of variables.
- In a full tree, every path from the root to a leaf contains all variables. The leaves correspond to possible worlds.

Example search tree

Suppose we have 3 variables, X with domain $\{a, b\}$, Y with domain $\{t, f\}$, and Z with domain $\{a, b, c\}$:



Basic Search Algorithm

$Q := \{\langle\rangle\}$;
$W := \{\}$;
While $Q \neq \{\}$ do
  choose and remove $\langle Y_1{=}v_1, \cdots, Y_j{=}v_j\rangle$ from Q;
  if $j = n$
    $W \leftarrow W \cup \{\langle Y_1{=}v_1, \cdots, Y_j{=}v_j\rangle\}$
  else
    Select a variable $Y_{j+1} \notin \{Y_1, \ldots, Y_n\}$
    $Q \leftarrow Q \cup \{\langle Y_1{=}v_1, \cdots, Y_j{=}v_j, Y_{j+1}{=}v\rangle : v \in dom(Y_{j+1})\}$

Q is a set of paths from root to a leaf.
W is a set of generated possible worlds.

Properties of the Algorithm

- Each partial description can only be generated once. There is no need to check for multiple paths or loops in the search.
- The probability of a world W is

$$\prod_i P(X_i | parents(X_i))_W$$

- Once a factor is fully assigned, we can multiply by its value.

Estimating the Probabilities

Use W, at the start of an iteration of the while loop, as an approximation to the set of all possible worlds.
Let

$$P_W^g = \sum_{w \in W \wedge w \models g} P(w)$$

$$P_Q = 1 - P_W^{true}$$

Then

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Then

$$P_W^g \leq P(g) \leq P_W^g + P_Q$$

Posterior Probabilities

Given the definition of conditional probability:

$$P(g|obs) = \frac{P(g \wedge obs)}{P(obs)}$$

We estimate the probability of a conditional probability:

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$$\frac{P_W^{g \wedge obs}}{P_W^{obs} + P_Q} \leq P(g|obs) \leq \frac{P_W^{g \wedge obs} + P_Q}{P_W^{obs} + P_Q}$$

If we choose the midpoint as an estimate:

$$\text{Error} \quad \leq$$

Posterior Probabilities

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If we choose the midpoint as an estimate:

$$\text{Error} \quad \leq \quad \frac{P_Q}{2(P_W^{obs} + P_Q)}$$

As the computation progresses, the probability mass in the queue P_Q approaches zero.

Refinements

- We only need to consider the ancestors of the variables we are interested in. We can prune the rest before the search.
- When computing $P(\alpha)$, we prune partial descriptions if it can be determined whether α is true or false in that partial description.
- When computing $P(\bullet|OBS)$, we prune partial descriptions in which OBS is false.
- We want to generate the most likely possible worlds to minimize the error. One good search strategy is a depth-first search, pruning unlikely worlds.

Recursive Conditioning

- Consider a factor graph where the nodes are factors and there are arcs between two factors that have a variables in common.
- Assigning a value v to a variable X, simplifies all factors that contain X.
  Factor F that contains X becomes factor $F_{X=v}$ which doesn't contain X.
- If an assignment disconnects the graph, each component can be evaluated separately.
- Computed values can be cached. The cache can be checked before evaluating any query.

Recursive Conditioning

procedure $rc(Fs:$ set of factors$)$:
  if $Fs = \{\}$ return 1
  else if $\exists v$ such that $\langle Fs, v\rangle \in cache$
    return v
  else if $\exists F \in Fs$ such that $vars(F) = \{\}$
    return $F \times rc(Fs \setminus F)$
  else if $Fs = Fs_1 \uplus Fs_2$ such that $vars(Fs_1) \cap vars(Fs_2) = \{\}$
    return $rc(Fs_1) \times rc(Fs_2)$
  else select variable $X \in vars(Fs)$
    $sum \leftarrow 0$
    for each $v \in dom(X)$
      $sum \leftarrow sum + rc(\{F_{X=v} : F \in Fs\})$
    $cache \leftarrow cache \cup \{\langle Fs, sum\rangle\}$
    return sum

Notes on the $rc(Fs)$ algorithm

- *cache* is a global variable that contains sets of pairs. It is initially empty.
- $vars(F)$ returns the unassigned variables in F
- $F_{X=v}$ is F with variable X assigned to value v
- $Fs = Fs_1 \uplus Fs_2$ is the disjoint union, meaning $Fs_1 \neq \{\}$, $Fs_2 \neq \{\}$, $Fs_1 \cap Fs_2 = \{\}$, $Fs = Fs_1 \cup Fs_2$
  This step recognizes when the graph is disconnected.

Exploiting Structure in Recursive Conditioning

- How can we exploit determinism (zero probabilities)?
- How can we exploit context-specific independencies. E.g., if $P(X|Y=y, Z=z) = P(X|Y=y, Z=z')$ for a particular y and for all values z, z'?