

# Making Decisions Under Uncertainty

What an agent should do depends on:

- The agent's **ability** — what options are available to it.
- The agent's **beliefs** — the ways the world could be, given the agent's knowledge.  
Sensing updates the agent's beliefs.
- The agent's **preferences** — what the agent wants and tradeoffs when there are risks.

Decision theory specifies how to trade off the desirability and probabilities of the possible outcomes for competing actions.

# Decision Variables

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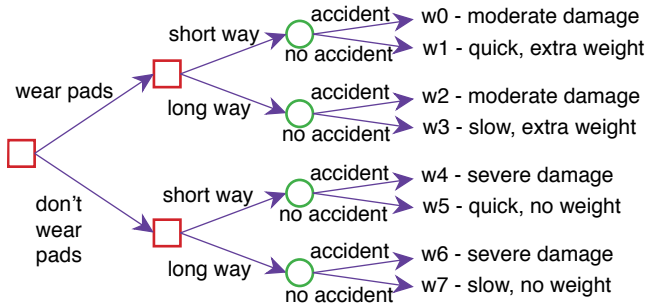
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- For each assignment of values to all decision variables, the measure of the set of worlds satisfying that assignment sum to 1.
- The probability of a proposition is undefined unless the agent conditions on the values of all decision variables.

# Decision Tree for Delivery Robot

The robot can choose to wear pads to protect itself or not.  
The robot can choose to go the short way past the stairs or a long way that reduces the chance of an accident.  
There is one random variable of whether there is an accident.



# Expected Values

- The expected value of a function of possible worlds is its average value, weighting possible worlds by their probability.
- Suppose  $f(\omega)$  is the value of function  $f$  on world  $\omega$ .
  - ▶ The **expected value** of  $f$  is

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- ▶ The **conditional expected value** of  $f$  given  $e$  is

$$\mathcal{E}(f|e) = \sum_{\omega|e} P(\omega|e) \times f(\omega).$$

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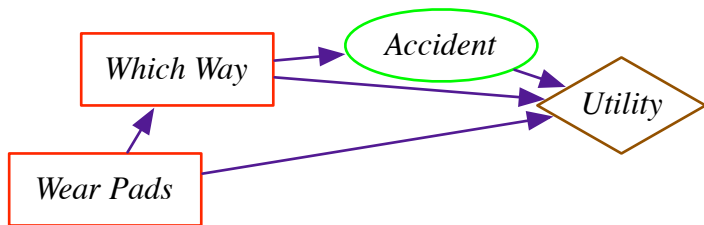
- Single decisions: agent makes all decisions before acting
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- An **optimal single decision** is a decision  $D = d_{max}$  whose expected utility is maximal:

$$\mathcal{E}(u|D = d_{max}) = \max_{d_i \in \text{dom}(D)} \mathcal{E}(u|D = d_i).$$

# Single-stage decision networks

Extend belief networks with:

- **Decision nodes**, that the agent chooses the value for. Domain is the set of possible actions. Drawn as rectangle.
- **Utility node**, the parents are the variables on which the utility depends. Drawn as a diamond.



This shows explicitly which nodes affect whether there is an accident.

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- utility function depends on the parents of the utility node
- no tables associated with the decision nodes

# Finding an optimal decision

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To find an optimal decision:

- ▶ Create a factor for each conditional probability and for the utility
- ▶ Sum out all of the random variables
- ▶ This creates a factor on  $D$  that gives the expected utility for each  $D$
- ▶ Choose the  $D$  with the maximum value in the factor.

# Example Initial Factors

Which Way	Accident	Value
long	true	0.01
long	false	0.99
short	true	0.2
short	false	0.8

Which Way	Accident	Wear Pads	Value
long	true	true	30
long	true	false	0
long	false	true	75
long	false	false	80
short	true	true	35
short	true	false	3
short	false	true	95
short	false	false	100

# After summing out Accident

Which Way	Wear Pads	Value
long	true	74.55
long	false	79.2
short	true	83.0
short	false	80.6



# Decision Networks

- flat or modular or hierarchical
- explicit states or features or individuals and relations
- static or finite stage or indefinite stage or infinite stage
- fully observable or partially observable
- deterministic or stochastic dynamics
- goals or complex preferences
- single agent or multiple agents
- knowledge is given or knowledge is learned
- perfect rationality or bounded rationality

# Sequential Decisions

- An intelligent agent doesn't carry out a multi-step plan ignoring information it receives between actions.
- A more typical scenario is where the agent: observes, acts, observes, acts, . . .
- Subsequent actions can depend on what is observed. What is observed depends on previous actions.
- Often the sole reason for carrying out an action is to provide information for future actions. For example: diagnostic tests, spying.

# Sequential decision problems

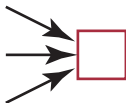
- A **sequential decision problem** consists of a sequence of decision variables  $D_1, \dots, D_n$ .
- Each  $D_i$  has an **information set** of variables  $parents(D_i)$ , whose value will be known at the time decision  $D_i$  is made.

# Decisions Networks

A **decision network** is a graphical representation of a finite sequential decision problem, with 3 types of nodes:



- A **random variable** is drawn as an ellipse. Arcs into the node represent probabilistic dependence.

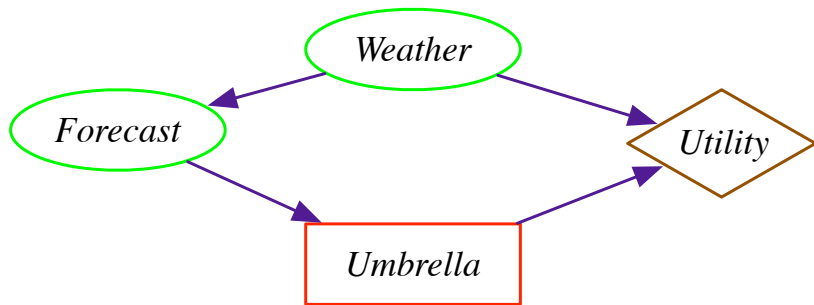


- A **decision variable** is drawn as a rectangle. Arcs into the node represent information available when the decision is made.



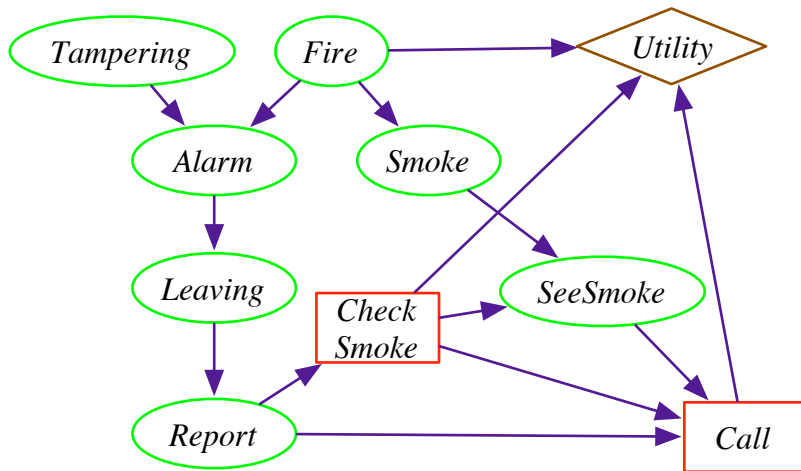
- A **utility** node is drawn as a diamond. Arcs into the node represent variables that the utility depends on.

# Umbrella Decision Network



You don't get to observe the weather when you have to decide whether to take your umbrella. You do get to observe the forecast.

# Decision Network for the Alarm Problem



- A **No-forgetting decision network** is a decision network where:
- The decision nodes are totally ordered. This is the order the actions will be taken.
  - All decision nodes that come before  $D_i$  are parents of decision node  $D_i$ . Thus the agent remembers its previous actions.
  - Any parent of a decision node is a parent of subsequent decision nodes. Thus the agent remembers its previous observations.

# What should an agent do?

- What an agent should do at any time depends on what it will do in the future.
- What an agent does in the future depends on what it did before.



- A policy specifies what an agent should do under each circumstance.
- A **policy** is a sequence  $\delta_1, \dots, \delta_n$  of **decision functions**

$$\delta_i : \text{dom}(\text{parents}(D_i)) \rightarrow \text{dom}(D_i).$$

This policy means that when the agent has observed  $O \in \text{dom}(\text{parents}(D_i))$ , it will do  $\delta_i(O)$ .

# Expected Utility of a Policy

- Possible world  $\omega$  **satisfies** policy  $\delta$ , written  $\omega \models \delta$  if the world assigns the value to each decision node that the policy specifies.
- The **expected utility of policy**  $\delta$  is

$$\mathcal{E}(u|\delta) = \sum_{\omega \models \delta} u(\omega) \times P(\omega),$$

- An **optimal policy** is one with the highest expected utility.

# Finding an optimal policy

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- Create a factor for each conditional probability table and a factor for the utility.
- Repeat:
  - ▶ Sum out random variables that are not parents of a decision node.
  - ▶ Let  $D$  be last variable
    - $D$  is only in a factor  $f$  with (some of) its parents.
  - ▶ Eliminate  $D$  by maximizing. This returns:
    - ▶ an optimal decision function for  $D$ :  $\arg \max_D f$
    - ▶ a new factor:  $\max_D f$
- until there are no more decision nodes.
- Sum out the remaining random variables. Multiply the factors: this is the expected utility of an optimal policy.

# Initial factors for the Umbrella Decision

Weather	Value
norain	0.7
rain	0.3

Weather	Fcast	Value
norain	sunny	0.7
norain	cloudy	0.2
norain	rainy	0.1
rain	sunny	0.15
rain	cloudy	0.25
rain	rainy	0.6

Weather	Umb	Value
norain	take	20
norain	leave	100
rain	take	70
rain	leave	0

# Eliminating By Maximizing

$f$ :

Fcast	Umb	Val
sunny	take	12.95
sunny	leave	49.0
cloudy	take	8.05
cloudy	leave	14.0
rainy	take	14.0
rainy	leave	7.0

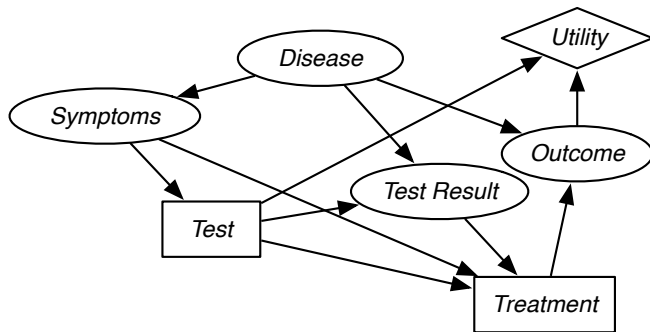
$\max_{Umb} f$ :

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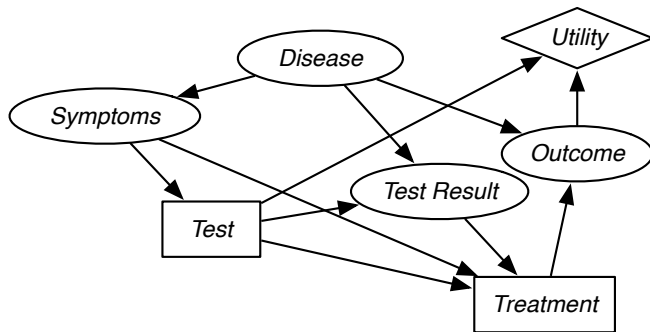
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# Exercise



What are the factors?

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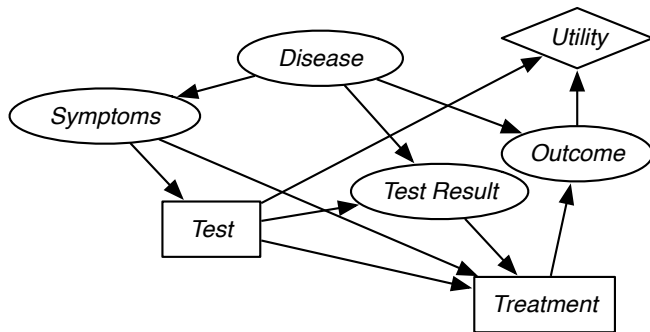


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Which random variables get summed out first?



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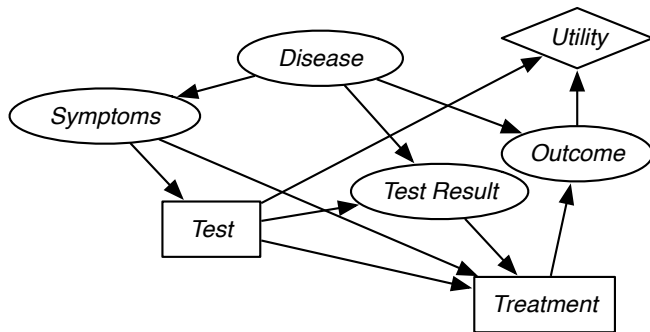


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Which decision variable is eliminated? What factor is created?

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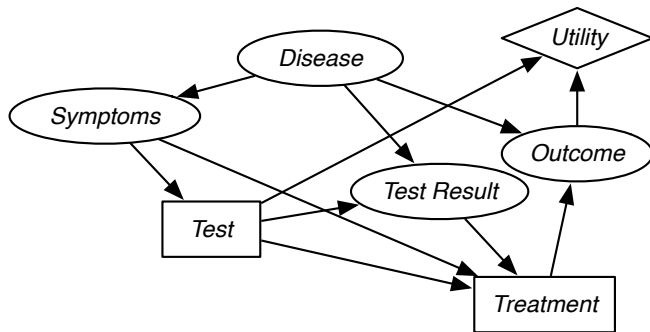
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Which decision variable is eliminated? What factor is created?

Then what is eliminated (and how)?

What factors are created after maximization?

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If there are multiple decision functions

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- The number of policies is the product of the number decision functions.
- The number of optimizations in the dynamic programming is the sum of the number of assignments of values to parents.
- The dynamic programming algorithm is much more efficient than searching through policy space.



# Value of Information

- The value of information  $X$  for decision  $D$  is the utility of the network with an arc from  $X$  to  $D$  (+ no-forgetting arcs) minus the utility of the network without the arc.
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- The value of information provides a bound on how much an agent should be prepared to pay for a sensor. How much is a better weather forecast worth?
- We need to be careful when adding an arc would create a cycle. E.g., how much would it be worth knowing whether the fire truck will arrive quickly when deciding whether to call them?

# Value of Control

- The value of control of a variable  $X$  is the value of the network when you make  $X$  a decision variable (and add no-forgetting arcs) minus the value of the network when  $X$  is a random variable.
- You need to be explicit about what information is available when you control  $X$ .
- If you control  $X$  without observing, controlling  $X$  can be worse than observing  $X$ . E.g., controlling a thermometer.
- If you keep the parents the same, the value of control is always non-negative.