

# Conditional independence

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i.e. for all  $\underline{x_i} \in \text{domain}(X)$ ,  $\underline{y_j} \in \text{domain}(Y)$ ,  $\underline{y_k} \in \text{domain}(Y)$  and  $\underline{z_m} \in \text{domain}(Z)$ ,

$$\begin{aligned} P(\underline{X = x_i} \mid Y = y_j \wedge Z = z_m) \\ &= P(\underline{X = x_i} \mid Y = y_k \wedge Z = z_m) \\ &= P(\underline{X = x_i} \mid Z = z_m). \end{aligned}$$

That is, knowledge of  $Y$ 's value doesn't affect the belief in the value of  $X$ , given a value of  $Z$ .

# Example

Consider a student writing an exam.

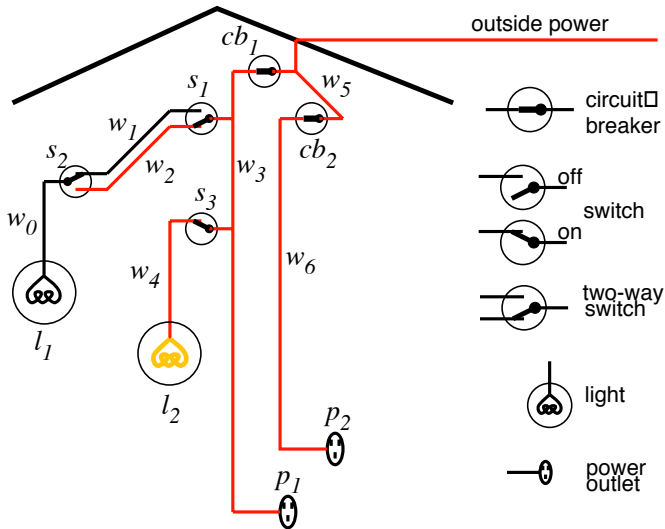
What are reasonable independences among the following?

- Whether the student works hard ( $W$ )
- Whether the student is intelligent ( $I$ )
- The student's answers on the exam ( $A$ )
- The student's mark on an exam ( $M$ )

reasonable to assume  
independent of  $T$

$M$  is independent of  $W$  given  $A$   
 $I$

# Example domain (diagnostic assistant)



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- Whether light  $l_1$  is lit is independent of the position of light switch  $s_2$  given



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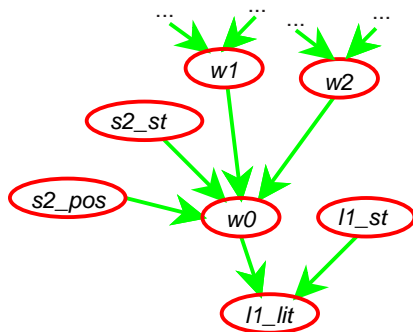
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- Whether light  $l_1$  is lit is independent of the position of light switch  $s_2$  given whether there is power in wire  $w_0$ .
- Every other variable may be independent of whether light  $l_1$  is lit given whether there is power in wire  $w_0$  and the status of light  $l_1$  (if it's *ok*, or if not, how it's broken).

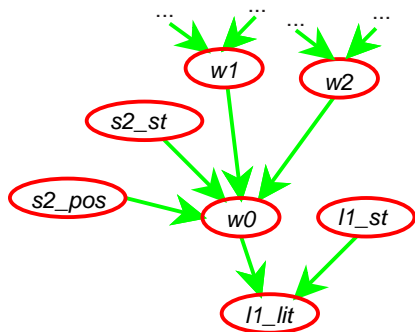
# Idea of belief networks

- $l_1$  is lit ( $L1\_lit$ ) depends only on the status of the light ( $L1\_st$ ) and whether there is power in wire  $w_0$ .
- In a belief network,  $W_0$  and  $L1\_st$  are **parents** of  $L1\_lit$ .
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- In a belief network,  $W_0$  and  $L1\_st$  are **parents** of  $L1\_lit$ .
- $W_0$  depends only on whether there is power in  $w_1$ , whether there is power in  $w_2$ , the position of switch  $s_2$  ( $S2\_pos$ ), and the status of switch  $s_2$  ( $S2\_st$ ).



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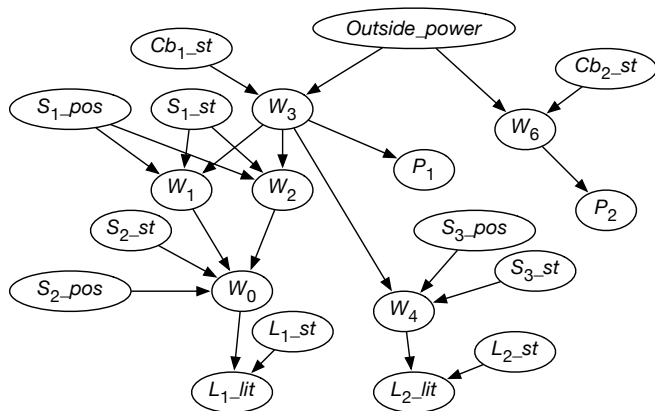
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- So  $P(X_1, \dots, X_n) = \prod_{i=1}^n P(X_i \mid parents(X_i))$
- A **belief network** is a graph: the nodes are random variables; there is an arc from the parents of each node into that node.

# Diagnosis Example

<http://aispace.org> → Downloads → Belief and Decision Networks → Load Sample Problem → Electrical Diagnosis Problem



# Student Writing an Exam Example

Give a belief network for the variables in order:

- *WorksHard*: Whether the student works hard
- *Intelligent*: Whether the student is intelligent
- *Answers*: The student's answers on the exam
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What if the variables were in the opposite order?

# Example: fire alarm belief network

## Variables:

- **Fire**: there is a fire in the building
- **Tampering**: someone has been tampering with the fire alarm
- **Smoke**: what appears to be smoke is coming from an upstairs window
- **Alarm**: the fire alarm goes off
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See “Fire Alarm Belief Network” in Alspace.org Belief and Decision Networks App

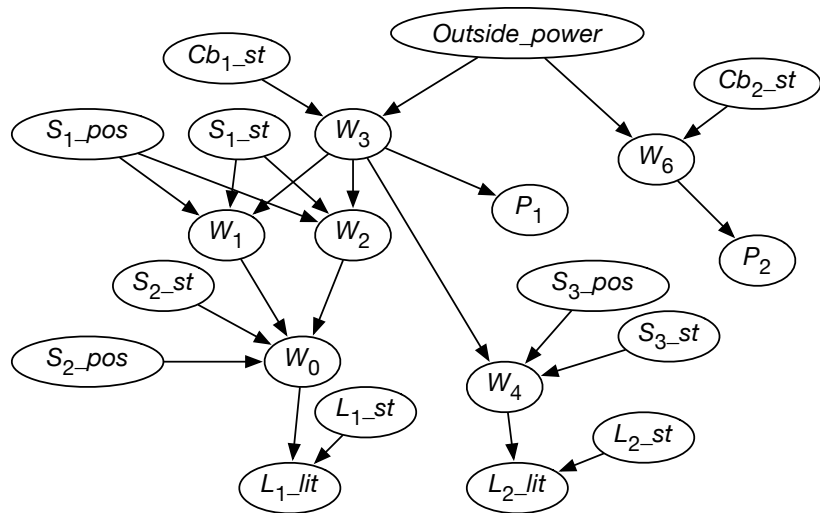


# Components of a belief network

A belief network consists of:

- a directed acyclic graph with nodes labeled with random variables
- a domain for each random variable
- a set of conditional probabilities, one for each variable given its parents (including prior probabilities for nodes with no parents).

# Example belief network



## Example belief network (continued)

The belief network also specifies:

- The domain of the variables:

$W_0, \dots, W_6$  have domain  $\{live, dead\}$

$S_{1\_pos}, S_{2\_pos},$  and  $S_{3\_pos}$  have domain  $\{up, down\}$

$S_{1\_st}$  has  $\{ok, upside\_down, short, intermittent, broken\}$ .

- Conditional probabilities, including:

$P(W_1 = live \mid s_{1\_pos} = up \wedge S_{1\_st} = ok \wedge W_3 = live)$

$P(W_1 = live \mid s_{1\_pos} = up \wedge S_{1\_st} = ok \wedge W_3 = dead)$

$P(S_{1\_pos} = up)$

$P(S_{1\_st} = upside\_down)$

# Belief network summary

- A belief network is a directed acyclic graph (DAG) where nodes are random variables.
- The **parents** of a node  $n$  are those variables on which  $n$  directly depends.
- A belief network is automatically acyclic by construction.
- A belief network is a graphical representation of dependence and independence:
  - ▶ A variable is independent of its non-descendants given its parents.

# Constructing belief networks

To represent a domain in a belief network, you need to consider:

- What are the relevant variables?
  - ▶ What will you observe?
  - ▶ What would you like to find out (query)?
  - ▶ What other features make the model simpler?
- What values should these variables take?
- What is the relationship between them? This should be expressed in terms of a directed graph, representing how each variable is generated from its predecessors.
- How does the value of each variable depend on its parents? This is expressed in terms of the conditional probabilities.

